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On the analytic continuation of solutions to nonlinear partial differential equations

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Abstract

We consider the analytic continuation of solutions to the nonlinear partial differential equation

$$\left(\frac{\partial}{\partial t}\right)^m u = F\left(t, x, \left\{\left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{\substack{j+|\alpha|\leqslant m\\ j\leqslant m-1}}\right)$$

in the complex domain. Suppose a solution $u(t,x)$ is known to be holomorphic in the domain $\{(t,x) \in \mathbb{C} \times \mathbb{C}^n;\ |x| < R,\ 0 < |t| < r \text{ and } |\arg t| < \theta\}$ for some positive numbers R, r and θ. Then we can show that if $u(t,x)$ satisfies some growth condition as t approaches zero, it is possible to extend it as a holomorphic solution of this partial differential equation up to some neighborhood of the origin.

Keywords: Partial differential equation; Analytic continuation

1. Introduction and main result

The investigation of the possibility of analytic continuation is an important problem in the theory of partial differential equations in the complex domain. In particular, in the study

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of singular solutions (i.e., solutions which possess some singularities) to partial differential equations, one way of arguing the nonexistence of such solutions is by means of analytic continuation.

If the partial differential equation is linear, then we have the well-known theorem of Zerner [4] in 1971 which states that any holomorphic solution may be extended analytically over noncharacteristic hypersurfaces. If the equation is not linear, then we have some results by Tsuno [3] in 1975 that attempt to extend those of Zerner. As may be expected, the nonlinear case is more difficult than the linear case, and thus Tsuno had to assume the boundedness of the solution and its derivatives in order to establish the possibility of analytic continuation. More than two decades later, Kobayashi [1] published in 1998 a more precise result on this problem. He formulated two possible premises to replace the boundedness assumption of Tsuno. The two conditions are not equivalent; one implies the other. We are of the opinion that one condition is relatively simpler than the other, but this condition has the disadvantage that it gives a less precise result.

This paper presents yet another result on this problem. We will come up with a precise result using as our premise what we deem is the simpler of the two conditions.

Let us now begin the formulation of the problem. Denote by $\mathbb{N}$ the set of all nonnegative integers and by $\mathbb{N}^*$ the set $\mathbb{N}\setminus\{0\}$. Let $m\in\mathbb{N}^*$, $n\in\mathbb{N}$ and Λ be the set of multi-indices:

$$\big\{(j,\alpha)\in\mathbb{N}\times\mathbb{N}^n;\ j+|\alpha|\leqslant m,\ j<m\big\}.$$

Let $(t,x)=(t,x_1,\dots,x_n)\in\mathbb{C}\times\mathbb{C}^n$ and consider the nonlinear partial differential equation:

$$\left(\frac{\partial}{\partial t}\right)^m u=F\left(t,x,\left\{\left(\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{(j,\alpha)\in\Lambda}\right). \tag{1.1}$$

All throughout this paper, we will assume that the function

$$F(t,x,Z)=F\big(t,x,(Z_{j,\alpha})_{(j,\alpha)\in\Lambda}\big)$$

is holomorphic in the domain $G\times H\times\mathbb{C}^{\#\Lambda}$, where $G=\{t\in\mathbb{C};\ |t|<r_0\}$ and

$$H=\big\{x\in\mathbb{C}^n;\ |x|<R_0\big\}$$

for some positive numbers r_0 and R_0. For any $\varepsilon>0$, we set $G_\varepsilon=\{t\in G\setminus\{0\};\ |\arg t|<\varepsilon\}$.

Now suppose that a solution $u(t,x)$ is known to be holomorphic in $G_\theta\times H$ for some $\theta>0$. We wish to answer the following question: *Under what conditions will it be possible to extend the solution $u(t,x)$ as a holomorphic solution of* (1.1) *up to some neighborhood of the origin?* We will answer this by focusing on the growth of $u(t,x)$ as t approaches the origin.

Since the function $F(t,x,Z)$ is holomorphic, we may expand it into the following convergent power series:

$$F(t,x,Z)=\sum_{\mu\in\mathcal{M}}a_\mu(t,x)Z^\mu=\sum_{\mu\in\mathcal{M}}t^{k_\mu}b_\mu(t,x)Z^\mu. \tag{1.2}$$

In the summation above, the set $\mathcal{M}$ has elements of the form $\mu = (\mu_{j,\alpha})_{(j,\alpha)\in\Lambda}$ and is a subset of $\mathbb{N}^{\#\Lambda}$; we agree to exclude from $\mathcal{M}$ those multi-indices μ for which the coefficient $a_\mu(t,x)$ is identically zero. The expression Z^μ is to be interpreted as the product $\prod_{(j,\alpha)\in\Lambda}(Z_{j,\alpha})^{\mu_{j,\alpha}}$. Moreover, we have taken out the maximum power of t from each $a_\mu(t,x)$, so that we have $b_\mu(0,x)\not\equiv 0$ for all $\mu\in\mathcal{M}$. Using this expansion, we can now write our partial differential equation as

$$\left(\frac{\partial}{\partial t}\right)^m u = \sum_{\mu\in\mathcal{M}} t^{k_\mu} b_\mu(t,x) \prod_{(j,\alpha)\in\Lambda}\left[\left(\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right]^{\mu_{j,\alpha}}. \tag{1.3}$$

Denote by $\gamma_t(\mu)$ the total number of derivatives with respect to t in the product

$$\prod_{(j,\alpha)\in\Lambda}\left[\left(\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right]^{\mu_{j,\alpha}},$$

that is,

$$\gamma_t(\mu) = \sum_{(j,\alpha)\in\Lambda} j\cdot\mu_{j,\alpha}. \tag{1.4}$$

Since the highest order of differentiation with respect to t appearing on the right-hand side is at most $m-1$, we have $\gamma_t(\mu)\leqslant(m-1)|\mu|$.

For any real number λ, we define:

$$\delta(\lambda) = \inf_{\mu\in\mathcal{M},|\mu|\geqslant 2}\big(k_\mu + m - \gamma_t(\mu) + \lambda\big(|\mu|-1\big)\big). \tag{1.5}$$

Kobayashi used this quantity in the hypothesis of his theorem. He obtained the following result:

Theorem 1 (Kobayashi, 1998). *Suppose that a solution $u(t,x)$ is known to be holomorphic in $G_\theta\times H$ and satisfies the estimate*:

$$\big\|u(t)\big\|_H = \sup_{x\in H}\big|u(t,x)\big| = \mathrm{O}\big(|t|^\sigma\big)\quad (\textit{as } t\to 0 \textit{ in } G_\theta). \tag{1.6}$$

If for this σ, we have $\delta(\sigma)>0$, then this solution may be extended as a holomorphic solution of (1.1) *up to some neighborhood of the origin.*

If $\delta(\sigma)$ is positive for some values of σ, then it is natural to think of the least σ for which $\delta(\sigma)>0$. Kobayashi then identified a critical value for σ, which he defined by:

$$\sigma_{\mathsf{K}} = \sup_{\mu\in\mathcal{M},\,|\mu|\geqslant 2}\frac{-k_\mu - m + \gamma_t(\mu)}{|\mu|-1}. \tag{1.7}$$

Since k_μ is nonnegative and $\gamma_t(\mu) \leqslant (m-1)|\mu|$, then it follows from the above definition that $\sigma_{\mathsf{K}} \leqslant m-1$. It may also be shown using the definition that $\delta(\sigma) \geqslant 0$ if and only if $\sigma \geqslant \sigma_{\mathsf{K}}$, and that $\sigma > \sigma_{\mathsf{K}}$ implies $\delta(\sigma) > 0$, but not the other way around. This last observation leads to the following corollary to Kobayashi's theorem.

Corollary 2 (Kobayashi, 1998). *Suppose that a solution $u(t,x)$ is known to be holomorphic in $G_\theta \times H$ and satisfies the estimate*:

$$\|u(t)\|_H = \sup_{x\in H}\bigl|u(t,x)\bigr| = \mathrm{O}\bigl(|t|^\sigma\bigr) \quad (\textit{as } t \to 0 \textit{ in } G_\theta). \tag{1.8}$$

If σ is strictly greater than σ_{K}, then this solution may be extended as a holomorphic solution of (1.1) *up to some neighborhood of the origin.*

The statement above is more straightforward and for us is more desirable than Theorem 1. Kobayashi himself might have preferred this to the preceding theorem, had there been no gap between the conditions $\sigma > \sigma_{\mathsf{K}}$ and $\delta(\sigma) > 0$. For the condition $\sigma > \sigma_{\mathsf{K}}$ actually yields a weaker result, as may be seen in the following example. For simplicity, let $(t,x) \in \mathbb{C}^2$ and consider the first-order nonlinear equation:

$$\frac{\partial u}{\partial t} = \mathrm{e}^u \frac{\partial u}{\partial x} = \left(\sum_{j=0}^{\infty} \frac{u^j}{j!}\right) \frac{\partial u}{\partial x}. \tag{1.9}$$

For this equation, we have $k_\mu = 0$ for all μ. It can be easily checked that $\delta(0) = 1$ and $\sigma_{\mathsf{K}} = \lim_{j\to\infty} -1/j = 0$. Note that Corollary 2 fails to guarantee the analytic continuation of a solution $u(t,x)$ satisfying $\|u(t)\|_H = \mathrm{O}(1)$ (as $t \to 0$ in G_θ). But Theorem 1 does, since $\delta(0)$ is positive!

We are therefore faced with a dilemma: the condition $\delta(\sigma) > 0$ yields a sharp result but is not as straightforward as the condition $\sigma > \sigma_{\mathsf{K}}$.

This paper resolves this dilemma. Our theorem gives up the first condition in favor of the second but comes up with the same degree of accuracy in the result.

Define the subset $\mathcal{M}_0$ of $\mathcal{M}$ by:

$$\mathcal{M}_0 = \bigl\{\mu \in \mathcal{M};\ |\mu| \geqslant 2 \text{ and } k_\mu + m - \gamma_t(\mu) + \sigma_{\mathsf{K}}\bigl(|\mu| - 1\bigr) = 0\bigr\}.$$

Then our result may be stated as follows:

Theorem 3. *Suppose that a solution $u(t,x)$ is known to be holomorphic in the domain $G_\theta \times H$. Then this solution may be extended as a holomorphic solution of* (1.1) *up to some neighborhood of the origin if any of the following two conditions is satisfied*:

(i) *The set $\mathcal{M}_0$ is empty and $\|u(t)\|_H = \mathrm{O}(|t|^{\sigma_{\mathsf{K}}})$ (as $t \to 0$ in G_θ).*
(ii) *The set $\mathcal{M}_0$ is not empty and $\|u(t)\|_H = \mathrm{o}(|t|^{\sigma_{\mathsf{K}}})$ (as $t \to 0$ in G_θ).*

Note that when $\mathcal{M}_0 = \emptyset$, then the quantity $k_\mu + m - \gamma_t(\mu) + \sigma_{\mathsf{K}}(|\mu| - 1)$ is strictly positive for all $|\mu| \geqslant 2$. Statement (i) of Theorem 3 says that if this is the case, then analytic continuation is possible whenever $\sigma \geqslant \sigma_{\mathsf{K}}$ (or equivalently, whenever $\delta(\sigma) \geqslant 0$). This in fact suggests that the condition $\delta(\sigma) > 0$ of Theorem 1 is not really optimal. On the other hand, statement (ii) guarantees that when $\mathcal{M}_0 \neq \emptyset$, then analytic continuation is possible whenever $\sigma > \sigma_{\mathsf{K}}$ (or equivalently, whenever $\delta(\sigma) > 0$).

Let us recall Eq. (1.9). Since $-1/j \neq 0 = \sigma_{\mathsf{K}}$ for all j, the set $\mathcal{M}_0$ is empty. Our theorem asserts that analytic continuation is possible whenever $\sigma \geqslant \sigma_{\mathsf{K}} = 0$. This agrees with the result of Theorem 1.

The growth condition assumed in (ii) above may not be weakened, say by assuming that we only have $\|u(t)\|_H = \mathrm{O}(|t|^{\sigma_{\mathsf{K}}})$ (as $t \to 0$ in G_θ). Consider the following nonlinear equation in two variables $(t, x) \in \mathbb{C}^2$:

$$\frac{\partial u}{\partial t} = u\left(\frac{\partial u}{\partial x}\right)^j \quad (j \in \mathbb{N}^*). \tag{1.10}$$

In this equation, $m = 1$ and $k_{(1,j)} = 0$, $\sigma_{\mathsf{K}} = -1/j$ and $\mathcal{M}_0$ is not empty. It may be verified that this equation has as a solution the function $u(t, x) = (-1/j)^{1/j}\, x t^{-1/j}$, which is of large order $|t|^{\sigma_{\mathsf{K}}}$. But clearly this has an essential discontinuity at $t = 0$. (For a more general treatment, the reader is referred to Section 3 of Kobayashi [1] which is devoted to the construction of singular solutions of order $|t|^{\sigma_{\mathsf{K}}}$.)

2. A family of majorant functions

Once again, the variables (t, x) will denote elements in $\mathbb{C} \times \mathbb{C}^n$. In the following discussion, we will use the following notations to describe majorant relations:

(i) If $a(x) = \sum a_\alpha x^\alpha$ and $A(x) = \sum A_\alpha x^\alpha$, then we say that $a(x) \ll A(x)$ if and only if for all $\alpha \in \mathbb{N}^n$, we have $|a_\alpha| \leqslant A_\alpha$.

(ii) If $g(t, x) = \sum g_{k,\alpha}(t - \varepsilon)^k x^\alpha$ and $G(t, x) = \sum G_{k,\alpha}(t - \varepsilon)^k x^\alpha$, then we say that $g(t, x) \ll_\varepsilon G(t, x)$ if and only if for all $(k, \alpha) \in \mathbb{N} \times \mathbb{N}^n$, we have $|g_{k,\alpha}| \leqslant G_{k,\alpha}$.

In 1953, Lax [2] made clever use of a certain majorant function to establish the convergence of a formal series. In proving our main result, we will be using a suitably modified version of Lax's function, defined as follows: for $z \in \mathbb{C}$ and $i \in \mathbb{N}$, we set:

$$\varphi_i(z) = \frac{1}{4S} \sum_{k=0}^{\infty} \frac{z^k}{(k+1)^{2+i}}. \tag{2.1}$$

Here, $S = 1 + 1/2^2 + 1/3^2 + \cdots = \pi^2/6$. This constant is introduced to facilitate computation.

Note that each $\varphi_i(z)$ converges for all $|z| < 1$ and thus defines a holomorphic function in this domain. Moreover, this family of functions satisfy a number of interesting majorant relations.

Proposition 4. *The following relations hold for the functions $\varphi_i(z)$:*

(a) $\varphi_0(z)\varphi_0(z) \ll \varphi_0(z)$;
(b) $\varphi_i(z) \ll \varphi_j(z)$ *for any* $i, j \in \mathbb{N}$ *with* $i > j$;
(c) $(1/2)^{2+i}\varphi_{i-1}(z) \ll \frac{\mathrm{d}}{\mathrm{d}z}\varphi_i(z) \ll \varphi_{i-1}(z)$ *for any* $i \in \mathbb{N}^*$;
(d) *Given any* $0 < \varepsilon < 1$, *there exists a constant* $C_{i,\varepsilon} > 0$ *such that*

$$\frac{1}{1-\varepsilon z}\varphi_i(z) \ll C_{i,\varepsilon}\varphi_i(z).$$

Proof. The first three relations may be easily verified using the definition of $\varphi_i(z)$. It may also be checked that $\varphi_i(z)\varphi_i(z) \ll 2^i\varphi_i(z)$ holds. Hence, to prove the fourth, it is sufficient to show that

$$\frac{1}{1-\varepsilon z} = \sum_{k=0}^{\infty} \varepsilon^k z^k \ll B_{i,\varepsilon}\varphi_i(z) \tag{2.2}$$

for some $B_{i,\varepsilon} > 0$. But this is the same as showing that for all k, we have:

$$4S\varepsilon^k(k+1)^{2+i} \leqslant B_{i,\varepsilon}$$

for some constant $B_{i,\varepsilon} > 0$. Since $\varepsilon^k(k+1)^{2+i}$ is close to zero for sufficiently large values of k, such constant exists. □

The following two lemmas will play important roles in the proof of the main theorem.

Lemma 5. *Let $f(x)$ be holomorphic and bounded by M in a neighborhood of $\{x \in \mathbb{C}^n;\ |x| \leqslant R_0\}$. Fix any positive $R < R_0$. Then there exists a constant $B_i > 0$, dependent on R but not on $f(x)$, such that*

$$f(x) \ll MB_i\varphi_i\left(\frac{x_1+\cdots+x_n}{R}\right).$$

Proof. We have

$$f(x) \ll \frac{M}{1-\frac{x_1+\cdots+x_n}{R_0}} \ll \frac{4SM}{1-\frac{x_1+\cdots+x_n}{R_0}}\varphi_i\left(\frac{x_1+\cdots+x_n}{R}\right), \tag{2.3}$$

since $4S\varphi_i(z) \gg 1$. Using (d) of Proposition 4 with $\varepsilon = R/R_0 < 1$, we obtain the desired result. □

Lemma 6. *Let $a(t,x)$ be holomorphic and bounded by A in a neighborhood of $\{(t,x) \in \mathbb{C}\times\mathbb{C}^n;\ |t| \leqslant r_0$ and $|x| \leqslant R_0\}$. We express $a(t,x)$ in the form $a(t,x) = t^q b(t,x)$,* *where $q \in \mathbb{N}$ and $b(0,x) \neq 0$. Now fix any $R < R_0$ and set $\varepsilon = cr/2$, where c is any number in $(0,1]$ and $r < r_0$ is sufficiently small. Then we have*:

$$a(t,x) \ll_\varepsilon 2Ac^q B_0 \varphi_0\left(\frac{t-\varepsilon}{cr} + \frac{x_1 + \cdots + x_n}{R}\right).$$

Here, the constant B_0 is the constant associated with φ_0 in the preceding lemma.

Proof. This lemma was essentially proved by Kobayashi in [1], but for the benefit of the reader, we will present a proof here.

For brevity, let us set $z = (t-\varepsilon)/cr + (x_1 + \cdots + x_n)/R$. We first note that t is majorized by

$$\begin{aligned} t &= \varepsilon + (t-\varepsilon) \ll_\varepsilon (\varepsilon + 4cr)\bigl(1 + (t-\varepsilon)/4cr\bigr) \\ &\ll_\varepsilon (\varepsilon + 4cr)4S\varphi_0(z). \end{aligned} \tag{2.4}$$

As for $b(t,x)$, we may expand it into $b(t,x) = \sum b_k(x)t^k$, where each $b_k(x)$ is holomorphic in a neighborhood of $\{x \in \mathbb{C}^n;\ |x| \leqslant R_0\}$ and satisfies

$$\bigl|b_k(x)\bigr| \leqslant \frac{A}{r_0^{q+k}}. \tag{2.5}$$

By Lemma 5, there exists a constant B_0 such that

$$b_k(x) \ll \frac{AB_0}{r_0^{q+k}}\varphi_0\left(\frac{x_1 + \cdots + x_n}{R}\right). \tag{2.6}$$

Combining this with (2.4) and setting $\varepsilon = cr/2$, we have:

$$\begin{aligned} a(t,x) &\ll_\varepsilon \sum_{k=0}^{\infty}\bigl[(\varepsilon + 4cr)4S\varphi_0(z)\bigr]^{q+k}\frac{AB_0}{r_0^{q+k}}\varphi_0(z) \\ &\ll_\varepsilon AB_0\varphi_0(z)\sum_{k=0}^{\infty}\left(\frac{18crS}{r_0}\right)^{q+k}, \end{aligned} \tag{2.7}$$

since we know that $\varphi_0(z)\varphi_0(z) \ll_\varepsilon \varphi_0(z)$. We finish off the proof by taking the term c^q out of the summation and fixing a sufficiently small $r > 0$ such that $18rS < r_0/2$. $\square$

3. Proof of main result

We will construct a holomorphic function $w(t,x)$ which coincides with $u(t,x)$ in an open set in $G_\theta \times H$, and show that this $w(t,x)$ is holomorphic in a domain containing the origin $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x^n$. The approach being used in this section is a sharp modification of the one by Kobayashi [1].

We consider the following initial value problem:

$$\begin{cases} \left(\dfrac{\partial}{\partial t}\right)^m w = \displaystyle\sum_{\mu \in \mathcal{M}} t^{k_\mu} b_\mu(t,x) \prod_{(j,\alpha)\in\Lambda} \left[\left(\dfrac{\partial}{\partial t}\right)^j \left(\dfrac{\partial}{\partial x}\right)^\alpha w\right]^{\mu_{j,\alpha}}, \\ \left(\dfrac{\partial}{\partial t}\right)^p w\bigg|_{t=\varepsilon} = \dfrac{\partial^p u}{\partial t^p}(\varepsilon, x), \quad 0 \leqslant p \leqslant m-1. \end{cases} \tag{3.1}$$

By the Cauchy–Kowalevsky Theorem for nonlinear equations, this initial value problem has a unique holomorphic solution $w(t,x)$, and by construction, $w(t,x)$ coincides with $u(t,x)$ in some neighborhood of $(\varepsilon, 0) \in \mathbb{C}_t \times \mathbb{C}_x^n$. We now have to show that the $w(t,x)$ we have found is holomorphic up to some neighborhood of the origin, i.e., we will show that the domain of convergence of the formal solution $w(t,x) = \sum_{k=0}^{\infty} w_k(x)(t-\varepsilon)^k$ contains the origin.

As it is too complicated to establish convergence by just working on the formal solution, we will instead construct a majorant function $W(t,x)$ for $w(t,x)$ that is, again, holomorphic in a neighborhood of the origin. The rest of the following discussion will be devoted to this task.

We note that since the function $F(t,x,Z)$ is holomorphic in $G \times H \times \mathbb{C}^{\#\Lambda}$, the expansion

$$F(t,x,Z) = \sum_{\mu\in\mathcal{M}} a_\mu(t,x) Z^\mu = \sum_{\mu\in\mathcal{M}} t^{k_\mu} b_\mu(t,x) Z^\mu \tag{3.2}$$

is valid in a neighborhood of the set

$$\Omega_\rho = G \times H \times \left\{ Z = (Z_{j,\alpha})_{(j,\alpha)\in\Lambda} \in \mathbb{C}^{\#\Lambda};\ |Z_{j,\alpha}| \leqslant \rho \text{ for all } (j,\alpha) \in \Lambda \right\}$$

for any positive ρ. Let M_ρ be a bound for $F(t,x,Z)$ in this neighborhood. Then in $G \times H$, the estimate $|t^{k_\mu} b_\mu(t,x)| \leqslant M_\rho/\rho^{|\mu|}$ holds, and hence by Lemma 6, we have:

$$t^{k_\mu} b_\mu(t,x) \ll_\varepsilon \frac{2M_\rho B_0}{\rho^{|\mu|}} c^{k_\mu} \varphi_0\left(\frac{t-\varepsilon}{cr} + \frac{x_1 + \cdots + x_n}{R}\right), \tag{3.3}$$

where $R \in (0, R_0)$ is fixed, c moves in the interval $(0,1]$, $r \in (0, r_0)$ is chosen to be small enough and fixed, and we have set $\varepsilon = cr/2$. Having fixed R and r, we can only play with the remaining unfixed constant c.

At this point, the discussion will have to branch, depending on whether the set $\mathcal{M}_0$ is empty or not.

Proof of (i) of Theorem 3 (*The case when* $M_0 = \emptyset$). Since $u(t,x) = \mathrm{O}(|t|^{\sigma\kappa})$ as $t \to 0$ in G_θ, by shrinking G_θ into $G_{\theta'}$ with $\theta' < \theta$ if necessary, we may assume that for

$1 \leqslant p \leqslant m-1$, we have $(\partial/\partial t)^p u(t,x) = \mathrm{O}(|t|^{\sigma\kappa - p})$ as $t \to 0$ in $G_{\theta'}$. This implies that there exist constants $L_p > 0$ such that

$$\sup_{|x| \leqslant R} \left| \frac{\partial^p u}{\partial t^p}(\varepsilon, x) \right| \leqslant L_p \varepsilon^{\sigma\kappa - p} \quad (0 \leqslant p \leqslant m-1). \tag{3.4}$$

(Note that $\varepsilon = cr/2$ is small enough since r may be chosen to be very small.) Applying Lemma 5 gives:

$$\frac{\partial^p u}{\partial t^p}(\varepsilon, x) \ll L_p \, \varepsilon^{\sigma\kappa - p} B_{m-p} \varphi_{m-p}\left(\frac{x_1 + \cdots + x_n}{R} \right). \tag{3.5}$$

Observe that we have chosen different functions to majorize the derivatives (at $t = \varepsilon$) of the solution $u(t,x)$.

With (3.5) and (3.3) in mind, we set up the following problem:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} \right)^m W \gg_\varepsilon \displaystyle\sum_{\mu \in \mathcal{M}} \frac{2B_0 M_\rho}{\rho^{|\mu|}} c^{k_\mu} \varphi_0(z) \prod_{(j,\alpha) \in \Lambda} \left[\left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha W \right]^{\mu_{j,\alpha}}, \\ \left(\dfrac{\partial}{\partial t} \right)^p W \bigg|_{t=\varepsilon} \gg \dfrac{\partial^p u}{\partial t^p}(\varepsilon, x), \quad 0 \leqslant p \leqslant m-1. \end{cases} \tag{M}$$

Here, for brevity, we have again set $z = (t-\varepsilon)/cr + (x_1 + \cdots + x_n)/R$. It is easily checked that any $W(t,x)$ satisfying the majorant relations above must majorize the solution $w(t,x)$ of (3.1).

We claim that we can construct one such $W(t,x)$ in the form

$$W(t,x) = L\varepsilon^{\sigma\kappa} B_m \varphi_m(z), \tag{3.6}$$

where the constants L and c will later be specified.

Let us first check the initial conditions. We have:

$$W(\varepsilon, x) \;=\; L\varepsilon^{\sigma\kappa} B_m \varphi_m\left(\frac{x_1 + \cdots + x_n}{R} \right) \tag{3.7}$$

and

$$\begin{aligned} \left(\frac{\partial}{\partial t} \right)^p W \bigg|_{t=\varepsilon} \;&=\; \frac{L\varepsilon^{\sigma\kappa} B_m}{(cr)^p} \varphi_m^{(p)}\left(\frac{x_1 + \cdots + x_n}{R} \right) \\ &\gg\; \frac{L\varepsilon^{\sigma\kappa - p} B_m}{2^{k(p,m)}} \varphi_{m-p}\left(\frac{x_1 + \cdots + x_n}{R} \right). \end{aligned} \tag{3.8}$$

The quantity $k(p,m)$ is the constant resulting from repeated applications of Proposition 4 (c). Comparing these with (3.5), we see that the initial conditions are satisfied if we choose L to satisfy:

$$L \geqslant \max_{0 \leqslant p \leqslant m-1} \left\{ 2^{k(p,m)} L_p B_{m-p} / B_m \right\}. \tag{3.9}$$

We choose and fix one such L.

Having already checked the initial conditions, we now consider the majorant relation involving $(\partial/\partial t)^m W(t,x)$. Computing in the same manner as had been done in checking the initial conditions, and setting ε equal to $cr/2$, we get:

$$\left(\frac{\partial}{\partial t}\right)^m W \gg_\varepsilon \frac{LB_m(r/2)^{\sigma\kappa-m}}{2^{k(m,m)}} c^{\sigma\kappa-m}\varphi_0(z). \tag{3.10}$$

Let us turn to the right-hand side. Using Proposition 4, we obtain the following majorant relations:

$$\frac{\partial W}{\partial t} = \frac{L\varepsilon^{\sigma\kappa B_m}}{cr}\varphi_m'(z) \ll_\varepsilon \frac{L\varepsilon^{\sigma\kappa} B_m}{cr}\varphi_{m-1}(z) \tag{3.11}$$

and

$$\frac{\partial W}{\partial x} = \frac{L\varepsilon^{\sigma\kappa} B_m}{R}\varphi_m'(z) \ll_\varepsilon \frac{L\varepsilon^{\sigma\kappa} B_m}{R}\varphi_{m-1}(z). \tag{3.12}$$

Combining these two gives:

$$\left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha W \ll_\varepsilon \frac{L\varepsilon^{\sigma\kappa} B_m}{(cr)^j R^{|\alpha|}}\varphi_{m-(j+|\alpha|)}(z). \tag{3.13}$$

Thus, the right-hand side (RHS) is majorized by:

$$\begin{aligned}
\text{RHS} &\ll_\varepsilon \sum_{\mu\in\mathcal{M}} \frac{2B_0M_\rho}{\rho^{|\mu|}} c^{k_\mu}\varphi_0(z) \prod_{(j,\alpha)\in\Lambda} \left\{\frac{L\varepsilon^{\sigma\kappa} B_m}{(cr)^j R^{|\alpha|}}\varphi_{m-(j+|\alpha|)}(z)\right\}^{\mu_{j,\alpha}} \\
&\ll_\varepsilon 2B_0M_\rho\varphi_0(z) \sum_{\mu\in\mathcal{M}} c^{k_\mu} \prod_{(j,\alpha)\in\Lambda} \left\{\frac{L\varepsilon^{\sigma\kappa} B_m}{(cr)^j R^{|\alpha|}\rho}\right\}^{\mu_{j,\alpha}} \\
&= 2B_0M_\rho\varphi_0(z) \sum_{\mu\in\mathcal{M}} c^{k_\mu+\sigma\kappa|\mu|-\gamma_t(\mu)} \prod_{(j,\alpha)\in\Lambda} \left\{\frac{LB_m(r/2)^{\sigma\kappa}}{r^j R^{|\alpha|}\rho}\right\}^{\mu_{j,\alpha}}.
\end{aligned} \tag{3.14}$$

In the simplifications above, we have used (a) and (b) of Proposition 4 as well as the fact that ε has been set equal to $cr/2$.

Let us wrap up this part of the computation. Comparing the right-hand side of (3.10) and the last line of (3.14), we can see that the first of the majorant relations in (M) is satisfied by $W(t,x) = L\varepsilon^{\sigma\kappa} B_m\varphi_m(z)$ if we can force the following inequality to hold:

$$\begin{aligned}
&\frac{LB_m(r/2)^{\sigma\kappa-m}}{2^{k(m,m)}(2B_0)} \\
&\quad\geqslant M_\rho \sum_{\mu\in\mathcal{M}} c^{k_\mu+m-\gamma_t(\mu)+\sigma\kappa(|\mu|-1)} \prod_{(j,\alpha)\in\Lambda} \left\{\frac{LB_m(r/2)^{\sigma\kappa}}{r^j R^{|\alpha|}\rho}\right\}^{\mu_{j,\alpha}}.
\end{aligned} \tag{3.15}$$

The expression on the left-hand side of the above inequality (which for convenience will be denoted by K) involves only fixed constants, while the right-hand side has constants c and ρ which we can vary as we please. Note also that M_ρ is dependent on ρ. Since $\mathcal{M}_0$ is empty, we know that

$$k_\mu + m - \gamma_t(\mu) + \sigma_{\mathsf{K}}\big(|\mu| - 1\big) > 0 \quad \text{for all } \mu \text{ with } |\mu| \geqslant 2.$$

If $|\mu| \leqslant 1$, then

$$\begin{aligned} &k_\mu + m - \gamma_t(\mu) + \sigma_{\mathsf{K}}\big(|\mu| - 1\big) \\ &\quad \geqslant k_\mu + 1 + (m - 1 - \sigma_{\mathsf{K}})\big(1 - |\mu|\big) \geqslant 1. \end{aligned} \tag{3.16}$$

Here we made use of the fact that $\gamma_t(\mu) \leqslant (m-1)|\mu|$ and that $\sigma_{\mathsf{K}} \leqslant m - 1$. Thus, for any $\mu \in \mathcal{M}$, we have $c^{k_\mu + m - \gamma_t(\mu) + \sigma_{\mathsf{K}}(|\mu| - 1)} \leqslant 1$.

As for the expression inside the brackets, we can choose and fix a large $\rho = \tilde{\rho}$ so that it becomes less than $1/2$. This fixes a value for M_ρ and makes the infinite series converge. We can therefore choose a number N large enough so that

$$M_{\tilde{\rho}} \sum_{\mu \in \mathcal{M}, |\mu| > N} c^{k_\mu + m - \gamma_t(\mu) + \sigma_{\mathsf{K}}(|\mu| - 1)} \left(\frac{1}{2}\right)^{|\mu|} < \frac{K}{2}. \tag{3.17}$$

To handle the remaining finite number of terms in the summation, we take the minimum power of c, that is, we let:

$$\nu = \min_{|\mu| \leqslant N} \big(k_\mu + m - \gamma_t(\mu) + \sigma_{\mathsf{K}}\big(|\mu| - 1\big)\big).$$

Since $\nu > 0$ and since c may be made as close to zero as we please, we choose $c = \tilde{c}$ so that

$$\tilde{c}^\nu M_{\tilde{\rho}} \sum_{|\mu| \leqslant N} \left(\frac{1}{2}\right)^{|\mu|} < \frac{K}{2}. \tag{3.18}$$

To summarize, we were able to establish our claim that for suitable values of the constants R, r, ρ and c, the function $W(t, x)$ in (3.6) will satisfy the relations posed in (M). By our choice of ε, the origin $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_x^n$ lies within

$$\big\{|z| = \big|(t - \varepsilon)/cr + (x_1 + \cdots + x_n)/R\big| < 1\big\},$$

the domain of convergence of $W(t, x)$, and of course, also within the domain of convergence of the formal solution $w(t, x)$. This establishes (i) of Theorem 3.

Proof of (ii) of Theorem 3 (*The case when* $\mathcal{M}_0 \neq \emptyset$). We will follow the arguments of the previous case. Since $u(t, x) = \mathrm{o}(|t|^{\sigma_{\mathsf{K}}})$ as $t \to 0$ in G_θ, then $(\partial/\partial t)^p u(t, x) = \mathrm{o}(|t|^{\sigma_{\mathsf{K}} - p})$

as $t \to 0$ in $G_{\theta'}$ with $\theta' < \theta$. This means that there exist constants L_p and functions $\eta_p(t)$ tending to zero as $t \to 0$ in $G_{\theta'}$ such that

$$\sup_{|x| \leqslant R} \left| \frac{\partial^p u}{\partial t^p}(\varepsilon, x) \right| \leqslant L_p\, \varepsilon^{\sigma\kappa - p} \eta_p(\varepsilon) \quad (0 \leqslant p \leqslant m-1). \tag{3.19}$$

Without loss of generality, we may assume that for any $a > 0$, $t^a = \mathrm{O}(\eta_p(t))$. (For otherwise, we replace $\eta_p(t)$ by a function which tends to zero at a slower rate.) Again by Lemma 5, we have:

$$\frac{\partial^p u}{\partial t^p}(\varepsilon, x) \ll L_p\, \varepsilon^{\sigma\kappa - p} \eta_p(\varepsilon)\, B_{m-p} \varphi_{m-p}\left(\frac{x_1 + \cdots + x_n}{R}\right). \tag{3.20}$$

We wish to find a function $W(t,x)$ satisfying (M). We seek it in the form

$$W(t,x) = L\varepsilon^{\sigma\kappa} \eta(\varepsilon) B_m \varphi_m(z), \tag{3.21}$$

where the constant $L > 0$ is to be determined later, and we define the function $\eta(\varepsilon)$ by $\eta(\varepsilon) = \max\{\eta_0(\varepsilon), \eta_1(\varepsilon), \ldots, \eta_{m-1}(\varepsilon)\}$.

As before, we can check that $W(t,x)$ satisfies the initial conditions if we choose:

$$L \geqslant \max_{0 \leqslant p \leqslant m-1} \left\{ 2^{k(p,m)} L_p B_{m-p} / B_m \right\}. \tag{3.22}$$

We then continue following the previous arguments and arrive at the following inequality which must hold in order for $W(t,x)$ to satisfy the majorant relations in (M):

$$\begin{aligned} \frac{L B_m (r/2)^{\sigma\kappa - m}}{2^{k(m,m)}(2B_0)} \geqslant\ & M_\rho \sum_{\mu \in \mathcal{M} \setminus \mathcal{M}_0} c^{k_\mu + m - \gamma_t(\mu) + \sigma\kappa(|\mu|-1)} \eta(\varepsilon)^{|\mu|-1} \\ & \times \prod_{(j,\alpha) \in \Lambda} \left\{ \frac{L B_m (r/2)^{\sigma\kappa}}{r^j R^{|\alpha|} \rho} \right\}^{\mu_{j,\alpha}} \\ & + M_\rho \sum_{\mu \in \mathcal{M}_0} \eta(\varepsilon)^{|\mu|-1} \prod_{(j,\alpha) \in \Lambda} \left\{ \frac{L B_m (r/2)^{\sigma\kappa}}{r^j R^{|\alpha|} \rho} \right\}^{\mu_{j,\alpha}}. \end{aligned} \tag{3.23}$$

Note that we have split the summation into two. Both sums may be made to converge by choosing a large ρ. Note further, that in the first summation, we still have the expression $c^{k_\mu + m - \gamma_t(\mu) + \sigma\kappa(|\mu|-1)}$ but in the second, this expression is simply equal to 1.

Just like before, the first summation may be made as small as we want, except for the addend corresponding to $|\mu| = 0$. To deal with this, we recall that we required $\eta_p(t)$ to satisfy $t^a/\eta_p(t) \to 0$ as $t \to 0$. Hence the addend may be made small by choosing a small value for c. As for the second summation, we recall that $\mu \in \mathcal{M}_0$ implies that $|\mu| \geqslant 2$ and so we can factor out at least one $\eta(\varepsilon)$. This compensates for the absence of c in the second summation, and therefore, it can also be made arbitrarily small. This establishes (ii) of Theorem 3, and the theorem is now completely proved.

4. Some remarks

The results of Tsuno and Kobayashi may be deduced from our theorem. We first state the following useful tool.

Lemma 7. *Let p be a nonnegative integer and s be a real number with $s > -1$. Suppose that a function $u(t,x)$ is holomorphic in $G_\theta \times H$ and satisfies the following estimates*:

$$\left\| \left(\frac{\partial}{\partial t}\right)^j u(t) \right\|_H = \mathrm{O}(1) \quad (\textit{as } t \to 0 \textit{ in } G_\theta), \quad \textit{for } 0 \leqslant j \leqslant p, \tag{4.1a}$$

$$\left\| \left(\frac{\partial}{\partial t}\right)^{p+1} u(t) \right\|_H = \mathrm{O}\big(|t|^s\big) \quad \big[\textit{respectively } \mathrm{o}\big(|t|^s\big)\big] \quad (\textit{as } t \to 0 \textit{ in } G_\theta). \tag{4.1b}$$

Then there exist functions $g_0(x), g_1(x), \ldots, g_p(x)$ holomorphic in H and a function $w(t,x)$ holomorphic in $G_\theta \times H$ such that

$$u(t,x) = g_0(x) + g_1(x)t + \cdots + g_p(x)\frac{t^p}{p!} + w(t,x). \tag{4.2}$$

Moreover, $w(t,x)$ satisfies the estimate:

$$\big\|w(t)\big\|_H = \mathrm{O}\big(|t|^{s+p+1}\big) \quad \big[\textit{respectively } \mathrm{o}\big(|t|^{s+p+1}\big)\big] \quad (\textit{as } t \to 0 \textit{ in } G_\theta). \tag{4.3}$$

Proof. For $0 \leqslant i \leqslant p$, we set:

$$g_i(x) = \left(\frac{\partial}{\partial t}\right)^i u(t,x) - \int_0^t \left(\frac{\partial}{\partial s}\right)^{i+1} u(s,x)\,\mathrm{d}s. \tag{4.4}$$

Since $u(t,x)$ is holomorphic with respect to x and $(\partial/\partial s)^{i+1}u(s,x)$ is integrable on $\{s = ht \in \mathbb{C};\ 0 < h \leqslant 1\}$ for any $t \in G_\theta$, we are assured that each $g_i(x)$ is also holomorphic with respect to x. We may solve for $u(t,x)$ in the equation above and proceed inductively to arrive at the expansion in (4.2). The function $w(t,x)$ is given by:

$$w(t,x) = \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_p} \left(\frac{\partial}{\partial s}\right)^{p+1} u(s,x)\,\mathrm{d}s\,\mathrm{d}t_p \cdots \mathrm{d}t_1, \tag{4.5}$$

so that we easily obtain (4.3). □

Let us now turn to Kobayashi's result. He defined *boundedness of order σ* to describe the growth of $u(t,x)$ as t tends to zero. If $\sigma \leqslant 0$, his definition is essentially the same as $\|u(t)\|_H = \mathrm{O}(|t|^\sigma)$. However, if $\sigma > 0$, he defined it as follows:

$$\left\| \left(\frac{\partial}{\partial t}\right)^{j} u(t) \right\|_{H} = \begin{cases} \mathrm{O}(1), & \text{for } j = 0, 1, \ldots, [\![\sigma]\!]; \\ \mathrm{O}\big(|t|^{\sigma - j}\big), & \text{for } j = [\![\sigma]\!] + 1, \end{cases} \tag{4.6}$$

where $[\![\sigma]\!]$ denotes the greatest integer less than or equal to σ. To cover both positive and negative values of σ, his critical value had to be redefined as

$$\sigma_{\mathsf{K}}^{*} = \sup_{\substack{\mu \in \mathcal{M}, \\ \nu \leqslant \mu,\, |\nu| \geqslant 2}} \frac{-k_\mu - m + \gamma_t(\nu)}{|\nu| - 1}. \tag{4.7}$$

Here the notation $\nu \leqslant \mu$ means that $\nu_{j,\alpha} \leqslant \mu_{j,\alpha}$ for all $(j, \alpha) \in \Lambda$.

Let $u(t, x)$ be a holomorphic function in $G_\theta \times H$ and suppose that it is bounded of order σ with $\sigma > 0$. In view of (4.6), we may apply Lemma 7 to write $u = v + w$, where v is holomorphic in $G \times H$, while w is holomorphic in $G_\theta \times H$ and has a growth order of $\mathrm{O}(|t|^{\sigma})$ as t tends to zero in G_θ. Set:

$$\mathcal{M}_0^{*} = \big\{(\mu, \nu);\ \mu \in \mathcal{M},\ \nu \leqslant \mu,\ |\nu| \geqslant 2 \text{ and } k_\mu + m - \gamma_t(\nu) + \sigma_{\mathsf{K}}^{*}\big(|\nu| - 1\big) = 0\big\}.$$

Then Kobayashi's result may now be reformulated and improved as follows:

Theorem 8. *Let a solution $u(t, x)$ be holomorphic in $G_\theta \times H$ and suppose that it may be written in the form $u = v + w$, where v is holomorphic in $G \times H$ while w is holomorphic in $G_\theta \times H$. Then this solution may be extended as a holomorphic solution of* (1.1) *up to some neighborhood of the origin if any of the following two conditions is satisfied*:

(i) *The set $\mathcal{M}_0^{*}$ is empty and $\|w(t)\|_H = \mathrm{O}(|t|^{\sigma_{\mathsf{K}}^{*}})$ (as $t \to 0$ in G_θ).*
(ii) *The set $\mathcal{M}_0^{*}$ is not empty and $\|w(t)\|_H = \mathrm{o}(|t|^{\sigma_{\mathsf{K}}^{*}})$ (as $t \to 0$ in G_θ).*

Proof. Substituting this solution $u = v + w$ into (1.3) and considering w as the new unknown, we get:

$$\left(\frac{\partial}{\partial t}\right)^{m} w = v_m + \sum_{\mu \in \mathcal{M}} t^{k_\mu} b_\mu(t, x) \prod_{(j,\alpha) \in \Lambda} \left[\left(\frac{\partial}{\partial t}\right)^{j} \left(\frac{\partial}{\partial x}\right)^{\alpha} (w + v)\right]^{\mu_{j,\alpha}}. \tag{4.8}$$

In the equation above, $v_m = -(\partial/\partial t)^m v(t, x)$ and is a known holomorphic function. If we expand the rightmost expression, we will obtain addends of the form $\{(\partial/\partial t)^j (\partial/\partial x)^\alpha w\}^p \bar{v}^q$, where $\bar{v}$ is also a known holomorphic function and the exponents run through nonnegative values with $p + q = \mu_{j,\alpha}$. This means that our new partial differential equation really has the form:

$$\left(\frac{\partial}{\partial t}\right)^{m} w = v_m + \sum_{\substack{\mu \in \mathcal{M}, \\ \nu \leqslant \mu}} t^{k_\mu} b_\mu(t, x) t^{l_\nu} c_\nu(t, x) \prod_{(j,\alpha) \in \Lambda} \left[\left(\frac{\partial}{\partial t}\right)^{j} \left(\frac{\partial}{\partial x}\right)^{\alpha} w\right]^{\nu_{j,\alpha}}. \tag{4.9}$$

Here we have also taken out the maximum power of t from the functions that came about after the binomial had been expanded (and hence we may assume that if $c_\nu(t,x) \not\equiv 0$ then we have $c_\nu(0,x) \not\equiv 0$).

The critical value of σ for this equation is given by:

$$\sigma^*(v) = \sup_{\substack{\mu\in\mathcal{M},\\ \nu\leqslant\mu,\, |\nu|\geqslant 2,\, c_\nu(t,x)\not\equiv 0}} \frac{-k_\mu - l_\nu - m + \gamma_t(\nu)}{|\nu|-1}. \tag{4.10}$$

Now let:

$$\mathcal{M}_0^*(v) = \bigl\{(\mu,\nu);\ \mu\in\mathcal{M},\ \nu\leqslant\mu,\ |\nu|\geqslant 2,\ c_\nu(t,x)\not\equiv 0 \text{ and}\\ k_\mu + l_\nu + m - \gamma_t(\nu) + \sigma^*(v)(|\nu|-1) = 0\bigr\}.$$

Then by Theorem 3, we know that the solution $u(t,x)$ may be extended as a holomorphic solution of (1.1) up to some neighborhood of the origin if any of the following holds:

(i) the set $\mathcal{M}_0^*(v)$ is empty and $\|w(t)\|_H = \mathrm{O}(|t|^{\sigma^*(v)})$ (as $t\to 0$ in G_θ);
(ii) the set $\mathcal{M}_0^*(v)$ is not empty and $\|w(t)\|_H = \mathrm{o}(|t|^{\sigma^*(v)})$ (as $t\to 0$ in G_θ).

Since $\sigma^*(v)\leqslant\sigma_{\mathsf{K}}^*$, we immediately obtain Theorem 8(ii). Moreover, if $\sigma^*(v) = \sigma_{\mathsf{K}}^*$ and $\mathcal{M}_0^*$ is empty, we can show that the set $\mathcal{M}_0^*(v)$ must also be empty. This then gives us Theorem 8(i). □

Finally, we show how Tsuno's result may be proved using our theorem. His theorem is as follows:

Theorem 9 (Tsuno, 1975). *Let a solution $u(t,x)$ be holomorphic in $G_\theta\times H$. If $u(t,x)$ and all its derivatives with respect to t up to order $m-1$ is bounded in this domain, then it may be extended as a holomorphic solution of* (1.1) *up to some neighborhood of the origin.*

Proof. We use Lemma 7 again to write $u = v + w$, where v is holomorphic in $G\times H$, while w is holomorphic in $G_\theta\times H$ and has a growth order of $\mathrm{O}(|t|^{m-1})$ as t tends to zero in G_θ. The desired conclusion now follows easily from Theorem 8. For from the fact that $\gamma_t(\nu)\leqslant(m-1)|\nu|$, we have:

$$\frac{-k_\mu - m + \gamma_t(\nu)}{|\nu|-1} \leqslant m-1+\frac{-k_\mu-1}{|\nu|-1}, \tag{4.11}$$

implying that σ_{K}^* is at most $m-1$. If in case σ_{K}^* is equal to $m-1$, then the set $\mathcal{M}_0^*$ will have to be empty. □

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